

# AN APPLICATION OF THE PRÉKOPA-LEINDLER INEQUALITY AND SOBOLEV REGULARITY OF WEIGHTED BERGMAN PROJECTIONS

YUNUS E. ZEYTUNCU

**ABSTRACT.** We prove a general version of [Boa84, Theorem 4.1] to obtain Sobolev estimates for weighted Bergman projections on convex Reinhardt domains by using the Prékopa-Leindler inequality.

## 1. INTRODUCTION

**1.1. Setup.** Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and let  $\lambda$  be a positive continuous function on  $\Omega$ . Let  $L^2(\Omega, \lambda)$  denote the space of square integrable functions on  $\Omega$  with respect to the measure  $\lambda(z)dV(z)$ , where  $dV(z)$  denotes the Lebesgue measure on  $\mathbb{C}^n$ . We denote the corresponding norm and inner product by  $\|\cdot\|_{\Omega, \lambda}$  and  $\langle \cdot, \cdot \rangle_{\Omega, \lambda}$ , respectively. The subspace of square integrable holomorphic functions is denoted by  $L_a^2(\Omega, \lambda)$ . The restriction on  $\lambda$  guarantees that  $L_a^2(\Omega, \lambda)$  is a closed subspace and therefore the orthogonal projection operator (called the weighted Bergman projection)  $\mathbf{B}_\Omega^\lambda : L^2(\Omega, \lambda) \rightarrow L_a^2(\Omega, \lambda)$  exists (see [PW90]). It follows from the Riesz representation theorem that  $\mathbf{B}_\Omega^\lambda$  is an integral operator

$$\mathbf{B}_\Omega^\lambda f(z) = \int_\Omega B_\Omega^\lambda(z, w) f(w) \lambda(w) dV(w)$$

where the kernel  $B_\Omega^\lambda(z, w)$  is called the weighted Bergman kernel.

For any natural number  $k$ , the (weighted)  $L^2$ -Sobolev space  $W^k(\Omega, \lambda)$  is a subspace of  $L^2(\Omega, \lambda)$  with the norm defined by

$$\|f\|_{k, \lambda}^2 = \sum_{|\beta+\gamma| \leq k} \int_\Omega \left| \frac{\partial^{\beta+\gamma}}{\partial \bar{z}^\beta \partial z^\gamma} f(z) \right|^2 \lambda(z) dV(z).$$

The values of  $k$  for which  $\mathbf{B}_\Omega^\lambda$  is bounded on  $W^k(\Omega, \lambda)$  depend on the geometric and potential theoretic properties of the pair  $(\Omega, \lambda)$ . When  $\mathbf{B}_\Omega^\lambda$  is bounded on  $W^k(\Omega, \lambda)$  for all  $k \geq 0$ , we say  $\mathbf{B}_\Omega^\lambda$  is exactly regular. For the case  $\lambda \equiv 1$ , we refer to [BS99] for a comprehensive survey. For results in the weighted setting, we refer to [CL97, BG95, Lig89, CDM14, CDM15] and the references therein.

One of the well-developed cases is when  $\Omega$  is a bounded smooth Reinhardt domain and  $\lambda \equiv 1$ . In this case, the radial symmetry and smoothness of the boundary are sufficient to conclude exact regularity (see [Str86] for a proof and [Boa84] for two separate proofs). Under the additional convexity assumption, Boas gives an elementary proof of this fact in [Boa84]. The convexity condition on  $\Omega$  is used to establish a Brunn-Minkowski type inequality [Boa84,

---

2010 *Mathematics Subject Classification.* Primary 32A25, Secondary 32A36.

*Key words and phrases.* Weighted Bergman projection, Prékopa-Leindler inequality, Reinhardt domain.

This work was partially supported by a grant from the Simons Foundation (#353525), and also by a University of Michigan-Dearborn CASL Faculty Summer Research Grant.

Lemma 5.2]. It turns out that this inequality is a special case of the Prékopa's inequality in [Pré71, Inequality 1.5]. The remaining arguments in Boas' proof do not require convexity and do hold on general Reinhardt domains. Therefore, a more general version of the inequality [Boa84, Lemma 5.2] would lead to a weighted version of [Boa84, Theorem 4.1]. In this note, we show that the Prékopa-Leindler inequality [Gar02, Theorem 7.1] serves this purpose and we obtain a generalization in the weighted setting.

**1.2. Statement.** Let  $\Omega$  be a bounded smooth convex Reinhardt domain in  $\mathbb{C}^n$ . The weights that we consider are of the form  $\lambda(z) = f(-\rho(z))$ , where  $\rho$  is a smooth multi-radial (i.e.,  $\lambda(z_1, \dots, z_n) = \lambda(|z_1|, \dots, |z_n|)$ ) convex defining function for  $\Omega$  and  $f : [0, \infty) \rightarrow \mathbb{R}$  is a positive, differentiable, and decreasing function. We also assume that  $f(x)$  satisfies a convexity condition of the form,

$$(1) \quad \frac{\left(f\left(\frac{x+y}{2}\right)\right)^2}{f(x)f(y)} \geq \delta$$

for some  $\delta > 0$  and for all  $x, y \in [0, \infty)$ .

We note that the weights of the form above cover many of the settings in the literature. Indeed,

- if  $f \equiv 1$  then we obtain the unweighted setting in [Boa84],
- if  $\Omega$  is the unit disc in  $\mathbb{C}^1$ ,  $\rho(z) = |z|^2 - 1$ , and  $f(x) = x^a \exp\left(-\frac{b}{x^c}\right)$  for some  $a \geq 1, b \geq 0$ , and  $c \geq 0$ , then we obtain weights in [Dos04, Zey13],
- if  $\Omega$  is the unit ball in  $\mathbb{C}^n$ ,  $\rho(z) = \|z\|^2 - 1$ , and  $f(x) = x^a$  for some  $a \geq 1$ , then we are in the setting of [FR75, Bek82],
- if  $\Omega$  is the unit ball in  $\mathbb{C}^n$ ,  $\rho(z) = \|z\|^2 - 1$ , and  $f(x) = \exp\left(-\frac{1}{x}\right)$ , then we recover [ČZ15, Theorem 2].

Now we state the main result of this note.

**Theorem 1.** *Let  $\Omega$  be a bounded smooth convex Reinhardt domain in  $\mathbb{C}^n$  and  $\lambda(z) = f(-\rho(z))$  where  $f$  and  $\rho$  are as above. Then  $\mathbf{B}_\Omega^\lambda$  is bounded on  $W^k(\Omega, \lambda)$  for all  $k \in \mathbb{N}$ .*

The new ingredient in the proof of Theorem 2 is the Prékopa-Leindler inequality (stated below). Readers can find the history and more applications of this inequality in [Gar02].

**Theorem 2.** [Gar02, Theorem 7.1] *Let  $0 < t < 1$  and let  $f, g$ , and  $h$  be nonnegative integrable functions on  $\mathbb{R}^n$  satisfying*

$$(2) \quad h((1-t)x + ty) \geq f(x)^{1-t}g(y)^t$$

*for all  $x, y \in \mathbb{R}^n$ . Then*

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-t} \left(\int_{\mathbb{R}^n} g(x)dx\right)^t.$$

## 2. PROOF OF THEOREM 1

**2.1. Preliminaries.** First, we mention that on a bounded smooth convex Reinhardt domain one can always find a defining function  $\rho(z)$  that is smooth multi-radial and convex. Indeed, in [HM12], authors showed that on a smooth bounded convex domain, there exists a defining function that is convex in a neighborhood of the boundary. Such a function can be extended to a defining function that is convex on  $\Omega$  by using a gauge (Minkowski) function of the

domain. The rotational symmetry can be also achieved by defining the function first on the radial image\* of the domain.

Next, we note that the weights of the form above satisfy a convexity condition that will be needed when we invoke Theorem 2 below. Indeed, for  $x, y \in \Omega$  we consider the following ratio:

$$\frac{\left(\lambda\left(\frac{x+y}{2}\right)\right)^2}{\lambda(x)\lambda(y)} = \frac{\left(f\left(-\rho\left(\frac{x+y}{2}\right)\right)\right)^2}{f\left(-\rho(x)\right)f\left(-\rho(y)\right)}.$$

Since  $\rho$  is a convex function, we have

$$\rho\left(\frac{x+y}{2}\right) \leq \frac{\rho(x) + \rho(y)}{2}.$$

Furthermore, since  $f$  is decreasing and  $f$  satisfies (1) we obtain

$$\frac{\left(f\left(-\rho\left(\frac{x+y}{2}\right)\right)\right)^2}{f\left(-\rho(x)\right)f\left(-\rho(y)\right)} \geq \frac{\left(f\left(-\frac{\rho(x) + \rho(y)}{2}\right)\right)^2}{f\left(-\rho(x)\right)f\left(-\rho(y)\right)} \geq \delta > 0.$$

Therefore, we get

$$(3) \quad \frac{\left(\lambda\left(\frac{x+y}{2}\right)\right)^2}{\lambda(x)\lambda(y)} \geq \delta > 0$$

for all  $x, y \in \Omega$ .

Many parts of the proof have been already appeared in [Boa84, Zey13, ČZ15]; however, we repeat them here for completeness. The new ingredient (application of the Prékopa-Leindler inequality) is in the proof of Lemma 3.

For a multi-index  $\gamma$ ,  $d_\gamma$  denotes the  $L^2$ -norm of the monomial  $z^\gamma$ , that is

$$d_\gamma^2 = \int_{\Omega} |z^\gamma|^2 \lambda(z) dV(z).$$

The set of monomials  $\left\{\frac{z^\gamma}{d_\gamma}\right\}$  forms an orthonormal basis of  $L_a^2(\Omega, \lambda)$  and the Bergman kernel  $B_\Omega^\lambda(z, w)$  is given by the sum

$$B_\Omega^\lambda(z, w) = \sum_{\gamma} \frac{z^\gamma \overline{w^\gamma}}{d_\gamma^2}.$$

For  $j \in \mathbb{N}$ , let  $\mathbf{S}_j$  denote the truncation operator on  $L_a^2(\Omega, \lambda)$ ; i.e. for  $f(z) = \sum_{\alpha} f_{\alpha} z^{\alpha}$

$$\mathbf{S}_j f(z) = \sum_{|\alpha| \leq j} f_{\alpha} z^{\alpha}.$$

Note that  $\mathbf{S}_j$  is a bounded operator with operator norm 1. Furthermore, for any holomorphic function  $f(z)$  the truncation  $\mathbf{S}_j f(z)$  is a polynomial and therefore is in  $L_a^2(\Omega, \lambda)$ . If the norms of the truncations  $\mathbf{S}_j f(z)$  are uniformly bounded then  $f(z)$  is also in  $L_a^2(\Omega, \lambda)$ .

---

\*The radial image  $\mathcal{R} \subset \mathbb{R}_+^n$  of a Reinhardt domain  $\Omega \subset \mathbb{C}^n$  is the image of  $\Omega$  in the  $|z_1|, \dots, |z_n|$  variables in the real Euclidean space, i.e., the angular variables are suppressed.

**2.2. Proof of Theorem 1.** Our goal is to show that for a given multi-index  $|\beta| \leq k$  and  $f \in W^k(\Omega, \lambda)$ ,

$$(4) \quad \left\| \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\|_\lambda^2 \lesssim \|f\|_{k,\lambda}^2$$

where the constant is independent of  $f$ .

Using the  $\mathbf{S}_j$  operator above and the radial symmetry it is clear that

$$\left\| \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\|_\lambda^2 = \lim_{j \rightarrow \infty} \left\| \mathbf{S}_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\|_\lambda^2.$$

Therefore, it is enough to prove that

$$(5) \quad \left\| \mathbf{S}_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\|_\lambda^2 \lesssim \|f\|_{k,\lambda}^2$$

where the constant is independent of  $f$  and  $j$ . We will need the following integration by parts lemmas to obtain this inequality.

**Lemma 1.** *For a given multi-index  $\beta$  there exists a bounded operator  $M_\beta$  on  $L_a^2(\Omega, \lambda)$  such that*

$$\left\langle h, \frac{\partial^\beta}{\partial z^\beta} g \right\rangle_\lambda = \left\langle \frac{\partial^\beta}{\partial z^\beta} M_\beta h, g \right\rangle_\lambda$$

for all holomorphic polynomials  $h$  and  $g \in L_a^2(\Omega, \lambda)$ .

**Lemma 2.** *For a given multi-index  $\beta$  there exists a constant  $K_\beta$  such that*

$$\left| \left\langle \frac{\partial^\beta}{\partial z^\beta} h, f \right\rangle_\lambda \right| \leq K_\beta \|h\|_\lambda \|f\|_{|\beta|,\lambda}$$

for all  $f \in W^{|\beta|}(\Omega, \lambda)$  and all holomorphic polynomials  $h$ .

Assuming these two lemmas for now, we obtain (5) as follows. Let  $h \in L_a^2(\Omega, \lambda)$ ,

$$\left\| \mathbf{S}_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\|_\lambda^2 = \sup_{\|h\|_\lambda \leq 1} \left| \left\langle h, \mathbf{S}_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\rangle_\lambda \right|^2.$$

On the other hand,

$$\begin{aligned} \left| \left\langle h, \mathbf{S}_j \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\rangle_\lambda \right| &= \left| \left\langle \mathbf{S}_j h, \frac{\partial^\beta}{\partial z^\beta} \mathbf{B}_\Omega^\lambda f \right\rangle_\lambda \right| \\ &= \left| \left\langle \frac{\partial^\beta}{\partial z^\beta} M_\beta \mathbf{S}_j h, \mathbf{B}_\Omega^\lambda f \right\rangle_\lambda \right| \quad \text{by Lemma 1} \\ &= \left| \left\langle \frac{\partial^\beta}{\partial z^\beta} M_\beta \mathbf{S}_j h, f \right\rangle_\lambda \right| \quad \text{by self adjointness} \\ &\lesssim \|M_\beta \mathbf{S}_j h\|_\lambda \|f\|_{k,\lambda} \quad \text{by Lemma 2} \\ &\lesssim \|\mathbf{S}_j h\|_\lambda \|f\|_{k,\lambda} \quad \text{by Lemma 1} \\ &\leq \|h\|_\lambda \|f\|_{k,\lambda} \end{aligned}$$

where the constants are clearly independent of  $j$  and  $f$ . In order to finish the proof of Theorem 1, it remains to prove the lemmas above.

**2.3. Proof of Lemma 2.** The same lemma has appeared in [ČZ15, Lemma 21], which is in fact a rendition of [Str86, Lemma 2.1] and [Boa84, Lemma 6.1]. We go over the arguments briefly for completeness, and refer to [ČZ15] for more details.

If  $f$  is supported on a compact subset of  $\Omega$  then the estimate follows easily from the Bergman inequality. Hence we assume  $f$  is supported in a small neighborhood of the boundary of  $\Omega$ . Furthermore, we choose this neighborhood such that we can find smooth orthonormal vector fields  $L_1, \dots, L_n$  with the property that  $L_1, \dots, L_{n-1}$  and  $L_n + \overline{L_n}$  are tangent to the boundary of  $\Omega$ . As a consequence of the Cauchy-Riemann equations, we can write any derivative of a holomorphic polynomial  $h$  in terms of these vector fields; indeed, there exist  $c_{ij} \in C^\infty(\overline{\Omega})$  such that

$$\frac{\partial}{\partial z_j} h = \left( \sum_{i=1}^{n-1} c_{ij} L_i + c_{nj} (L_n + \overline{L_n}) \right) h =: \mathcal{L}_j h .$$

Also note that by the choice of  $\lambda = f(-\rho)$ ,  $T(\lambda) = 0$  for any tangential vector field  $T$ . This means, we have

$$\begin{aligned} \left\langle \frac{\partial}{\partial z_j} h, f \right\rangle_\lambda &= \langle \mathcal{L}_j(h), f \rangle_\lambda \quad \text{for some tangential vector field } \mathcal{L}_j \\ &= \langle \mathcal{L}_j(h)\lambda, f \rangle \\ &= \langle \mathcal{L}_j(h\lambda), f \rangle \quad \text{since } \mathcal{L}_j(\lambda) = 0 \\ &= \left\langle h\lambda, \widetilde{\mathcal{L}_j}(f) \right\rangle \quad \text{no boundary terms since } \mathcal{L}_j \text{ is tangential} \\ &= \left\langle h, \widetilde{\mathcal{L}_j}(f) \right\rangle_\lambda \end{aligned}$$

where  $\widetilde{\mathcal{L}_j}$  is a first order differential operator with  $C^\infty(\overline{\Omega})$  coefficients. For a multi-index  $\beta$ , if we iterate this argument  $|\beta|$  times we get,

$$\left\langle \frac{\partial^\beta}{\partial z^\beta} h, f \right\rangle_\lambda = \left\langle h, \widetilde{\mathcal{L}_\beta}(f) \right\rangle_\lambda$$

for some differential operator  $\widetilde{\mathcal{L}_\beta}$  of order  $|\beta|$  with  $C^\infty(\overline{\Omega})$  coefficients. We conclude the proof of Lemma 2 by the Hölder's inequality.

**2.4. Proof of Lemma 1.** This lemma is stated in [Boa84, Lemma 4.2] for  $\lambda \equiv 1$ . We define  $M_\beta$  as follows. For a monomial  $z^\alpha$ , we set

$$M_\beta(z^\alpha) = \frac{(\alpha + \beta)! (\alpha + \beta)!}{\alpha! (\alpha + 2\beta)!} \frac{d_\alpha^2}{d_{\alpha+\beta}^2} z^{\alpha+2\beta} .$$

The explicit expression of  $M_\beta$  is imposed by the orthonormality of the set  $\left\{ \frac{z^\gamma}{d_\gamma} \right\}$ , and the point of the lemma is to prove the continuity of  $M_\beta$ .

For this purpose, we compute the norms

$$\frac{\|M_\beta(z^\alpha)\|_\lambda}{\|z^\alpha\|_\lambda} = \frac{(\alpha + \beta)! (\alpha + \beta)!}{\alpha! (\alpha + 2\beta)!} \frac{d_\alpha d_{\alpha+2\beta}}{d_{\alpha+\beta}^2}$$

the first fraction is uniformly bounded since

$$\frac{(\alpha + \beta)!(\alpha + \beta)!}{\alpha!(\alpha + 2\beta)!} = \frac{\binom{\alpha + \beta}{\beta}}{\binom{\alpha + 2\beta}{\beta}} \leq 1.$$

It remains to prove that the second fraction

$$\frac{d_\alpha d_{\alpha+2\beta}}{d_{\alpha+\beta}^2}$$

is uniformly bounded. This is a consequence of the Prékopa-Liendler inequality as explained in the next lemma.

**Lemma 3.** *For a given multi-index  $\beta$  there exists a constant  $K_\beta$  such that*

$$(6) \quad d_\alpha d_{\alpha+2\beta} \leq K_\beta (d_{\alpha+\beta})^2$$

*for all multi-indices  $\alpha$ .*

*Proof.* We want to show that there exists  $K_\beta > 0$  such that

$$(7) \quad \int_{\Omega} |z^\alpha|^2 \lambda(z) dV(z) \int_{\Omega} |z^{\alpha+2\beta}|^2 \lambda(z) dV(z) \leq K_\beta \left( \int_{\Omega} |z^{\alpha+\beta}|^2 \lambda(z) dV(z) \right)^2.$$

Recall that  $\Omega$  is a Reinhardt domain and all the functions inside the integrals are multi-radial; therefore, the integration is taking place on the radial image  $\mathcal{R}$  of  $\Omega$ . We use  $r$  to denote the vector in  $\mathbb{R}^n$ . We define three functions

$$h(r) = r^{\zeta+\eta} \lambda(r) \chi_{\mathcal{R}}(r)$$

$$f(r) = r^\zeta \lambda(r) \chi_{\mathcal{R}}(r)$$

$$g(r) = r^{\zeta+2\eta} \lambda(r) \chi_{\mathcal{R}}(r)$$

where  $\zeta$  and  $\eta$  are multi-indices and  $\chi_{\mathcal{R}}(r)$  stands for the characteristic function of the convex set  $\mathcal{R}$ . We verify the condition (2) in Theorem 2. Namely, we claim

$$K_\eta h\left(\frac{x+y}{2}\right) \geq \sqrt{f(x)g(y)}$$

for all  $x, y \in \mathcal{R}$  and for some  $K_\eta > 0$ . That is, we claim

$$K_\eta \left(\frac{x+y}{2}\right)^{\zeta+\eta} \lambda\left(\frac{x+y}{2}\right) \chi_{\mathcal{R}}\left(\frac{x+y}{2}\right) \geq \sqrt{x^\zeta y^{\zeta+2\eta} \lambda(x) \lambda(y) \chi_{\mathcal{R}}(x) \chi_{\mathcal{R}}(y)}.$$

We can eliminate  $\chi_{\mathcal{R}}$ 's from both sides since the set  $\mathcal{R}$  is convex. We can also eliminate  $\lambda$ 's from both sides by the assumption (3). Therefore it remains to show that

$$K_\eta \left(\frac{x+y}{2}\right)^{\zeta+\eta} \geq \sqrt{x^\zeta y^{\zeta+2\eta}}.$$

Furthermore, it is enough to prove this in one dimension since the multi dimensional version follows by iteration.

In other words, we claim that for  $u, v \geq 0$  and  $a, b \in \mathbb{N}$ , there exists  $K > 0$  (independent of  $a$ ) such that

$$K \left(\frac{u+v}{2}\right)^{a+b} \geq \sqrt{u^a v^{a+2b}}.$$

We rewrite this inequality as follows

$$\begin{aligned} K \left( \frac{u+v}{2} \right)^{a+b} &\geq \sqrt{u^a v^{a+2b}} \\ K \left( \frac{u+v}{2} \right)^a \left( \frac{u+v}{2} \right)^b &\geq \sqrt{u^a v^{a+2b}} \\ K \left( \frac{\frac{\sqrt{u}}{\sqrt{v}} + \frac{\sqrt{v}}{\sqrt{u}}}{2} \right)^a \left( \frac{\frac{u}{v} + 1}{2} \right)^b &\geq 1 \end{aligned}$$

If we set  $t = \frac{\sqrt{u}}{\sqrt{v}}$  and observe that  $\frac{t+t^{-1}}{2} \geq 1$  on  $(0, \infty)$ , then we conclude the desired inequality by choosing  $K = 2^b$ .

This means we can invoke Theorem 2 for the functions  $h, f, g$  and appropriate multi-indices  $\zeta$  and  $\eta$  to obtain (7). This concludes the proof of Lemma 3 and the proof of Theorem 1.  $\square$

*Remark.* In the unweighted setting (i.e.  $\lambda \equiv 1$ ) the exact regularity holds without any convexity or pseudoconvexity assumption as demonstrated in [Str86]. Therefore, it is plausible to think that in the weighted setting, the radial symmetry of the domain and weight should be sufficient to conclude boundedness on Sobolev spaces. However, our proof of Theorem 1 does require convexity of the domain and weight, since we invoke the Prékopa-Leindler inequality. It will be a curious project to investigate whether one can drop the convexity (or even the pseudoconvexity) assumption on  $\Omega$  and obtain regularity for arbitrary weights.

#### ACKNOWLEDGMENT

I'd like to thank anonymous referees for constructive comments. I'd also like to thank Bo Berndtsson for his enlightening lectures on complex Brunn-Minkowski theory at the Analysis and Geometry in Several Complex Variables Conference at Texas A&M University at Qatar. I'd also like to thank the organizers of this conference, Shiferaw Berhanu, Nordine Mir and Emil J. Straube for a memorable time in Qatar.

#### REFERENCES

- [Bek82] David Bekollé. Inégalité à poids pour le projecteur de Bergman dans la boule unité de  $\mathbf{C}^n$ . *Studia Math.*, 71(3):305–323, 1981/82.
- [BG95] Aline Bonami and Sandrine Grellier. Weighted Bergman projections in domains of finite type in  $\mathbf{C}^2$ . In *Harmonic analysis and operator theory (Caracas, 1994)*, volume 189 of *Contemp. Math.*, pages 65–80. Amer. Math. Soc., Providence, RI, 1995.
- [Boa84] Harold P. Boas. Holomorphic reproducing kernels in Reinhardt domains. *Pacific J. Math.*, 112(2):273–292, 1984.
- [BS99] Harold P. Boas and Emil J. Straube. Global regularity of the  $\bar{\partial}$ -Neumann problem: a survey of the  $L^2$ -Sobolev theory. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 79–111. Cambridge Univ. Press, Cambridge, 1999.
- [CDM14] P. Charpentier, Y. Dupain, and M. Mounkaila. Estimates for weighted Bergman projections on pseudo-convex domains of finite type in  $\mathbf{C}^n$ . *Complex Var. Elliptic Equ.*, 59(8):1070–1095, 2014.
- [CDM15] P. Charpentier, Y. Dupain, and M. Mounkaila. On estimates for weighted Bergman projections. *Proc. Amer. Math. Soc.*, 143(12):5337–5352, 2015.

- [CL97] Der-Chen Chang and Bao Qin Li. Sobolev and Lipschitz estimates for weighted Bergman projections. *Nagoya Math. J.*, 147:147–178, 1997.
- [ČZ15] Željko Čučković and Yunus E. Zeytuncu. Mapping properties of weighted Bergman projection operators on Reinhardt domains. *to appear in Proc. Amer. Math. Soc.*, 2015.
- [Dos04] Milutin R. Dostanić. Unboundedness of the Bergman projections on  $L^p$  spaces with exponential weights. *Proc. Edinb. Math. Soc. (2)*, 47(1):111–117, 2004.
- [FR75] Frank Forelli and Walter Rudin. Projections on spaces of holomorphic functions in balls. *Indiana Univ. Math. J.*, 24:593–602, 1974/75.
- [Gar02] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405, 2002.
- [HM12] A.-K. Herbig and J. D. McNeal. Convex defining functions for convex domains. *J. Geom. Anal.*, 22(2):433–454, 2012.
- [Lig89] Ewa Ligocka. On the Forelli-Rudin construction and weighted Bergman projections. *Studia Math.*, 94(3):257–272, 1989.
- [Pré71] András Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)*, 32:301–316, 1971.
- [PW90] Zbigniew Pasternak-Winiarski. On the dependence of the reproducing kernel on the weight of integration. *J. Funct. Anal.*, 94(1):110–134, 1990.
- [Str86] Emil J. Straube. Exact regularity of Bergman, Szegő and Sobolev space projections in nonpseudoconvex domains. *Math. Z.*, 192(1):117–128, 1986.
- [Zey13] Yunus E. Zeytuncu. Sobolev regularity of weighted Bergman projections on the unit disc. *Complex Var. Elliptic Equ.*, 58(3):309–315, 2013.

UNIVERSITY OF MICHIGAN-DEARBORN, DEPARTMENT OF MATHEMATICS AND STATISTICS, DEARBORN, MI 48128

*E-mail address:* zeytuncu@umich.edu